Signal Analysis and Measurement Techniques

Introduces Fourier methods of analysis permitting the analysis of vibroacoustic problems in the frequency domain

1. Fourier Methods In Vibroacoustics

Most machines emit periodic disturbances

Few examples are

- Meshing of gear teeth
- Imbalances in rotating shafts
- Periodic pressure fluctuations in IC engine cylinders

They consist of different frequency components and are analysed using Fourier Series expansion
Approximation of signals

- Study how to best approximate a signal \( a(t) \) with a signal \( b(t) \).
- Assume that the fitting of the two signals is to take place during the time interval \( 0 < t < T \).
- To carry out the approximation in the simplest possible way, multiply \( b(t) \) by a constant \( \beta \) that is varied to adjust the approximation as far as possible,

\[
a(t) \approx \beta b(t)
\]

The error in the our approximation then becomes

\[
e(t) = a(t) - \beta b(t)
\]

The averaged squared error is computed over the entire time interval \( 0 < t < T \),

\[
\varepsilon = \frac{1}{T} \int_{0}^{T} \left( a(t) - \beta b(t) \right)^2 dt
\]
We minimize \( \varepsilon \) with respect to \( \beta \) by differentiating, and setting the resulting derivative equal to zero,

\[
\frac{d \varepsilon}{d \beta} = \frac{1}{T} \int_{0}^{T} 2(\beta b^2(t) - a(t)b(t)) \, dt = 0
\]

from which

\[
\beta = \left( \int_{0}^{T} a(t)b(t) \, dt \right) / \left( \int_{0}^{T} b^2(t) \, dt \right)
\]

We now introduce the useful concept of orthogonality. If the signals \( a(t) \) and \( b(t) \) are orthogonal, there is no connection between the two signals, and \( \beta \) equals zero.

The orthogonality condition for the two signals \( a(t) \) and \( b(t) \) on the time interval \( 0 < t < T \) is therefore

\[
\int_{0}^{T} a(t)b(t) \, dt = 0
\]
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The energy of a signal is proportional to the time integral of the signal squared.

\[ E_a = \alpha \int_0^T a^2(t) \, dt \]  

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in which \( \alpha \) is a proportionality constant. Given a signal composed of two parts, \( v(t) = a(t) + b(t) \), its energy becomes

\[ E_v = \alpha \int_0^T (a(t) + b(t))^2 \, dt = \alpha \int_0^T a^2(t) \, dt + \alpha \int_0^T b^2(t) \, dt + \alpha \int_0^T 2a(t)b(t) \, dt \]  

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If the signals are orthogonal

\[ E_v = \alpha \int_0^T a^2(t) \, dt + \alpha \int_0^T b^2(t) \, dt = E_a + E_b \]  

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Thus, the energy of the combined signal is the sum of the individual energies.
Possibly, the approximation of the signal \( a(t) \) given by Eq. (1) in combination with Eq. (5) gives an inadequate fit. To further reduce the error, we incorporate another signal \( c(t) \) with a proportionality constant \( \gamma \), and write the approximation as

\[
a(t) \approx \beta b(t) + \gamma c(t)
\]  

\[\ldots10\]

The error in that case becomes

\[
e(t) = a(t) - \beta b(t) - \gamma c(t)
\]

\[\ldots11\]

and the mean squared error is

\[
\varepsilon = \frac{1}{T} \int_0^T (a(t) - \beta b(t) - \gamma c(t))^2 dt
\]

\[\ldots12\]

minimizing error \( \varepsilon \) with respect to \( \beta \)

\[
\frac{\partial \varepsilon}{\partial \beta} = \frac{1}{T} \int_0^T 2(\beta b^2(t) + \gamma b(t)c(t) - a(t)b(t)) dt = 0
\]

\[\ldots13\]
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If we can now choose \( b(t) \) and \( c(t) \) to be mutually orthogonal, the second term in equation (13) is eliminated

\[
\beta = \int_0^T a(t) b(t) dt \sqrt{\int_0^T b^2(t) dt}
\]

minimizing the error with respect to \( \gamma \)

\[
\frac{\partial \varepsilon}{\partial \gamma} = \frac{1}{T} \int_0^T 2(\gamma c^2(t) + \beta b(t)c(t) - a(t)c(t))dt = 0
\]

If \( b(t) \) and \( c(t) \) are orthogonal

\[
\gamma = \int_0^T a(t) c(t) dt \sqrt{\int_0^T c^2(t) dt}
\]

By comparison to Eq. (5), the result obtained is the same as what would have followed from the approximation \( a(t) = \gamma c(t) \). To improve upon that, we therefore only need to add another orthogonal signal and minimize the mean squared error versus \( a(t) \), independently of the other signals included in the approximation. Examples of functions that are orthogonal are \( \sin(nw_0 t) \) and \( \cos(nw_0 t) \).
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Fourier series decomposition

Assume a signal $a(t)$ that is periodic, with period $T$, and which we wish to approximate with the help of sine and cosine functions,

$$a(t) = \beta_0 + \sum_{n=1}^{\infty} \beta_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} \gamma_n \sin(n\omega_0 t) \quad ....17$$

That is called a Fourier series decomposition of the signal $a(t)$. The coefficients $\beta_n$ and $\gamma_n$ can be calculated separately, and given by Eq. (5)

$$\beta_0 = \frac{1}{T/2} \int_{-T/2}^{T/2} a(t)dt \quad \gamma_n = \frac{1}{T/2} \int_{-T/2}^{T/2} a(t) \sin(n\omega_0 t) dt \quad ....18$$
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\[
\beta_n = \frac{1}{T} \left[ \int_{-T/2}^{T/2} a(t) \cos(n\omega_0 t) dt \right] \int_{-T/2}^{T/2} \cos^2 (n\omega_0 t) dt = \frac{2}{T} \int_{-T/2}^{T/2} a(t) \cos(n\omega_0 t) dt, \quad n = 1, 2, 3, \ldots
\]

…..19

\[
\gamma_n = \frac{1}{T} \left[ \int_{-T/2}^{T/2} a(t) \sin(n\omega_0 t) dt \right] \int_{-T/2}^{T/2} \sin^2 (n\omega_0 t) dt = \frac{2}{T} \int_{-T/2}^{T/2} a(t) \sin(n\omega_0 t) dt, \quad n = 1, 2, 3, \ldots
\]

…..20

The interval of integration in Eqs. (18), (19) and (20) is \(-T/2\) to \(T/2\), but could just as well have been 0 to \(T\). The coefficient \(\beta_0\) represents the signal’s time average. It can also be shown that corresponding sine and cosine terms can be combined into a single cosine term with a phase angle \(\phi_n\).

\[
a(t) = \beta_0 + \sum_{n=1}^{\infty} \delta_n \cos(n\omega_0 t - \phi_n)
\]

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where
\[ \delta_n = \sqrt{\beta_n^2 + \gamma_n^2}, \quad \phi_n = \arctan\left(\frac{\gamma_n}{\beta_n}\right) + m \pi \]

where
- \( \delta_1 \) gives the signal’s amplitude for the first tone, or fundamental tone,
- \( \delta_2 \) gives the signal’s amplitude for the second tone, or the first overtone,
- \( \delta_3 \) gives the signal’s amplitude for the third tone, or second overtone.

\[ \text{(21)} \]
Figure 1 Cylinder pressure in a single-cylinder, two-cycle engine running at 5500 rpm.

a) The pressure as a function of the crankshaft angle $\phi$. The pressure variation repeats itself periodically after 360°, i.e., after a complete cycle. The different phases are compression, ignition, combustion, exhaust, and intake. For a two-cycle engine, exhaust and intake occur simultaneously, by virtue of the inflowing gases pushing out the exhaust gases.

b) The sound pressure level as a function of frequency. The periodic pressure variation, after Fourier series decomposition, results in a frequency spectrum with discrete frequency components. For a running speed of 5500 rpm, the fundamental frequency is $f_0 = 5500/60 = 91.7$ Hz.
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To denote each frequency component as a complex, rotating vector results in simpler computations and a more compact symbolic expression, a complex Fourier series can be defined,

\[ a(t) = \sum_{n=-\infty}^{\infty} \delta_n e^{i n \omega_0 t} \]  

…..22

where the complex coefficient \( \delta_n \) can be determined from

\[ \delta_n = \frac{1}{T/2} \int_{-T/2}^{T/2} a(t) e^{-in\omega_0 t} dt \]  

…..23
In Eq. (22), there are components with “negative frequencies”. That is because,
a real quantity can be described with the aid of two oppositely rotating complex
vectors, each of which is the complex conjugate of the other.

\[
\delta_n = \beta_n + i\gamma_n, \quad \delta_n^* = \beta_n - i\gamma_n
\]

**Figure 2** Three-dimensional description of the frequency spectrum of a periodic
signal. Complex Fourier series contain components with both negative and positive
frequencies. (Source: Brüel &Kjær, Frequency Analysis.)
How good an approximation of the original rectangular waveform that one obtains depends, naturally, on the number of terms included in the Fourier series decomposition.
Figure 3 Synthesis of a rectangular waveform with different numbers of terms in the Fourier series decomposition. The graph shows the gradual improvement of the Fourier series approximation when more Fourier components are included in the summation. In the intervals in which the original time function has continuous derivatives, the Fourier series approximation can be arbitrarily close to the original function with enough terms. For discontinuous points, such as those at $t/T = 0, 0.5, 1.0, 1.5, \text{ etc.}$, there is always a “ripple”, no matter how many terms are included in the approximation. That is called the Gibb’s phenomenon in mathematics. It can be shown that the size of the ripple is, at a maximum, about 18% of the size of the discontinuity.
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Fourier transform

Many machines or processes give rise to sound or vibrations which are not periodic, but rather random (stochastic) or transient. Roughness of the contacting surfaces between a wheel and its path, for instance, or between meshing gear teeth, bring about randomly varying vibrations. Turbulence in flowing media gives rise to randomly varying sound. For non-periodic disturbances, one must use a so-called Fourier transform.

Fourier transform can be developed from the complex Fourier series of a periodic train of pulses.

![Figure 4](image.png)

**Figure 4** Periodic train of pulses with amplitude 1, period $T$ and pulse-width $\alpha T$. 
Equation (23) provides the coefficients of the complex Fourier series,

\[
\delta_n = \frac{1}{T} \int_{-\alpha T/2}^{\alpha T/2} e^{-i\alpha \omega_0 t} dt = \frac{\left(e^{i\alpha \omega_0 T/2} - e^{-i\alpha \omega_0 T/2}\right)}{i\omega_0 T} = \frac{2i\sin(\alpha n \omega_0 T/2)}{i\omega_0 T} \quad \ldots 24
\]

but \( T = \frac{2\pi}{\omega_0} \), which implies that

\[
\delta_n = \alpha \frac{\sin(\alpha n \pi)}{\alpha n \pi} \quad \ldots 25
\]

Fourier series of the pulse train becomes

\[
F(t) = \sum_{n=-\infty}^{\infty} \alpha \frac{\sin(\alpha n \pi)}{\alpha n \pi} e^{i\alpha \omega_0 t} \quad \ldots 26
\]

Fourier series coefficients for the cases \( \alpha = \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \) are shown.
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![Graphs showing frequency components](image)

- For $\alpha = 1/2$, the frequency components are distributed symmetrically around the origin.
- For $\alpha = 1/4$, the frequency components are more concentrated around the origin, indicating a higher degree of coherence or periodicity in the signal.

These graphs illustrate the impact of different values of $\alpha$ on the frequency spectrum of a signal, demonstrating how varying $\alpha$ affects the distribution and magnitude of frequency components.
Figure 5  Fourier-series decomposition of a periodic pulse train with constant pulse width $\alpha T$. For smaller $\alpha$ and increasing period $T$, the frequency components become all the more densely packed a) $\alpha = 1/2$, b) $\alpha = 1/4$ and c) $\alpha = 1/8$. 

$$\alpha = \frac{1}{8}$$
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To derive a relation for a non-periodic event, we therefore consider the limiting case of the period $T$ becoming infinite.

If we substitute in the expression for the Fourier coefficients (23) into the Fourier series (22), we then obtain

$$ F(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i n \omega_0 t} \int_{-T/2}^{T/2} F(t) e^{-i n \omega_0 t} \, dt $$

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To adapt that to the limiting case when the period $T$ goes to infinity, the interchange $\omega_0 \to d\omega$ is made because $\omega_0 = \frac{2\pi}{T}$, and $n\omega_0$ transforms into a continuous variable $\omega$, i.e., $n\omega_0 \to \omega$. The step size in the summation becomes infinitesimally small, and the summation in equation (27) transforms to an integral

$$ F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i \omega_0 t} \left( \int_{-\infty}^{\infty} F(t) e^{-i \omega_0 t} \, dt \right) d\omega $$

...28
The expression inside the parentheses is identified as the Fourier transform of the signal,

$$F(\omega) = \int_{-\infty}^{\infty} F(t)e^{-i\omega t} dt \quad \ldots 29$$

and the inverse Fourier transform is given by

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{i\omega t} d\omega \quad \ldots 30$$

The Fourier transform is a complex quantity, which, in the case of $F(t)$ representing a force, has the units N/Hz. In order for $F(t)$ to be real, $F(-\omega) = F^*(\omega)$ must hold.
Parseval’s relation

Let $F_1(\omega)$ and $F_2(\omega)$ be Fourier transforms of the time functions $F_1(t)$ and $F_2(t)$.

$$\int_{-\infty}^{\infty} F_1(t)F_2(t)dt = \int_{-\infty}^{\infty} F_1(\omega)F_2^*(\omega)\frac{d\omega}{2\pi}$$

...31

which is called Parseval’s relation. To prove equation (31), put the inverse Fourier transform of $F_1(t)$ into the left hand side of equation (31)

$$\int_{-\infty}^{\infty} F_1(t)F_2(t)dt = \int_{-\infty}^{\infty} F_2(t) \int_{-\infty}^{\infty} F_1(\omega)e^{i\omega t} \frac{d\omega}{2\pi} dt =$$

$$= \int_{-\infty}^{\infty} F_1(\omega) \left[ \int_{-\infty}^{\infty} F_2(t)e^{i\omega t} dt \right] \frac{d\omega}{2\pi} =$$
Parseval’s relation can be used to calculate the mean square value of a quantity from a measured frequency spectrum.

\[ F^2 = \lim_{T \to \infty} \left( \frac{1}{T} \int_{-T/2}^{T/2} F^2(t) dt \right) \]  

...32

Parseval’s relation yields

\[ \int_{-\infty}^{\infty} F_1^2(t) dt = \int_{-\infty}^{\infty} F_1(\omega) F_1^*(\omega) \frac{d\omega}{2\pi} = \int_{-\infty}^{\infty} |F_1(\omega)|^2 \frac{d\omega}{2\pi} \]  

...33
Putting equation (33) into equation (32), and using the relation \( F(\omega) = F^*(-\omega) \), yields

\[
\tilde{F}^2 = \lim_{T \to \infty} \left( \frac{1}{T} \int_{-T/2}^{T/2} |F(t)|^2 \, dt \right) = \lim_{\Omega \to \infty} \left( \frac{1}{\Omega} \int_{-\Omega/2}^{\Omega/2} |F(\omega)|^2 \frac{d\omega}{2\pi} \right) = \lim_{\Omega \to \infty} \left( \frac{2}{\Omega} \int_{0}^{\Omega/2} |F(\omega)|^2 \frac{d\omega}{2\pi} \right)
\]

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Parseval’s relation for periodic signals can also be derived, as

\[
\int_{-T/2}^{T/2} F_1(t) F_2(t) \, dt = T \sum_{n = -\infty}^{\infty} d_{1n} d_{2n}^* \quad \ldots 35
\]

in which \( d_{1n} \) and \( d_{2n} \) are the Fourier series coefficients of \( F_1(t) \) and \( F_2(t) \), respectively.
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Using equation (22),

\[ \frac{T}{2} \int_{-T/2}^{T/2} F_1(t) F_2(t) dt = \frac{T}{2} \int_{-T/2}^{T/2} \left( \sum_{n=-\infty}^{\infty} d_{1n} e^{i\omega t} \sum_{m=-\infty}^{\infty} d_{2m}^{*} e^{-i\omega t} \right) dt = \]

\[ = \sum_{n=-\infty}^{\infty} d_{1n} \sum_{m=-\infty}^{\infty} d_{2m}^{*} \int_{-T/2}^{T/2} e^{i(n-m)\omega t} dt, \]

yet because

\[ \int_{-T/2}^{T/2} e^{i(n-m)\omega t} dt = \begin{cases} T & \text{f"{o}r } n = m \\ 0 & \text{f"{o}r } n \neq m \end{cases} \]

then it follows that

\[ \frac{T}{2} \int_{-T/2}^{T/2} F_1(t) F_2(t) dt = T \sum_{n=-\infty}^{\infty} d_{1n} d_{2n}^{*} \]
Equation (35) can be used to calculate the mean squared value of a periodic signal from its Fourier components

$$\tilde{F}^2 = \frac{1}{T} \int_{-T/2}^{T/2} F^2(t) \, dt = \sum_{n=-\infty}^{\infty} |d_n|^2$$  \hspace{1cm} \ldots 36$$

If the summation is only carried out over the “positive” frequencies, \( n = 0, 1, 2, \ldots \), then \( \tilde{F}_n^2 = 2|d_n|^2 \) must be used as the mean square value of the \( n \)-th frequency component. That follows from \( d_{-n} = d_n^* \) from which \( |d_{-n}| = |d_n| \)

That also provides the basis for computing the third octave band spectrum, for example, from a narrow band spectrum, or the octave band spectrum from a third-octave band spectrum.
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Filters

Frequency analysis means

- Study of a signal’s distribution along the frequency axis
- Two methods that are primarily used: the Fast Fourier Transform (FFT), and digital filtering

1. Low pass filter.
2. High pass filter
3. Band pass filter
4. Band stop filter

The filter type that is most common is the low pass filter. Such filters are often used at the input to a measurement system to filter away frequency components higher than those to be analyzed.
Figure 6 Different filters influence on a noise signal’s frequency spectrum when the signal passes through them: (i) Low pass filter (ii) High pass filter (iii) Band pass filter (iv) Band stop filter (Source: Brüel & Kjær, course material.)
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Band pass filters

The frequency components immediately outside of the pass band are not completely eliminated. A common way to define the upper $f_u$ and lower $f_l$ frequency limits of the band is to indicate the frequencies at which the signal is reduced by 3 dB.

\[
B = f_u - f_l
\]

Figure 7 a) Ideal band pass filter with infinitely steep cutoffs. b) Real filters have imperfect cutoffs. The upper and lower bounding frequencies are then defined by the frequencies at which the filter reduces the signal by 3 dB.
A filter with a bandwidth proportional to its center frequency, \( f_c \), is called a constant relative bandwidth (CRB) filter.

Figure 8a  CAB filter, with a bandwidth that does not vary along the frequency axis; it is typically presented with a linear frequency axis.
Figure 8b  CRB filter, with a bandwidth that is a certain percentage of the center frequency $f_c$; it is typically presented with a logarithmic scale. Because of the logarithmic scale, the stacks in the figure do not get wider, moving to the right along the axis. The example in the figure is called a third octave band filter and has a bandwidth that is about 23% of the center frequency.
Third-octave and octave band filters

Third-octave and octave band filters are CRB filters very widely used in the field of sound and vibrations. Center frequencies are standardized - Table 2

<table>
<thead>
<tr>
<th></th>
<th>Octave band filter</th>
<th>Third-octave band filter</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Lower limit</strong></td>
<td>$f_l = f_c / \sqrt{2}$</td>
<td>$f_l = f_c / \sqrt[6]{2}$</td>
</tr>
<tr>
<td><strong>Upper limit</strong></td>
<td>$f_u = \sqrt{2} f_c$</td>
<td>$f_u = \sqrt[6]{2} f_c$</td>
</tr>
<tr>
<td><strong>Bandwidth</strong></td>
<td>$B = f_u - f_l = (\sqrt{2} - 1/\sqrt{2})f_c$</td>
<td>$B = f_u - f_l = (\sqrt[6]{2} - 1/\sqrt[6]{2})f_c$</td>
</tr>
<tr>
<td><strong>Center frequency</strong></td>
<td>$f_c = \sqrt{f_l f_u}$</td>
<td>$f_c = \sqrt[6]{f_l f_u}$</td>
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</tbody>
</table>

Table 1 Definition of third-octave and octave band filters.
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Table 2  Standardized center frequencies and upper and lower frequency limits of third-octave and octave band filters. Shading indicates octave bands

<table>
<thead>
<tr>
<th>Band no.</th>
<th>Center frequency $f_c$ [Hz]</th>
<th>3rd-octave band filter $f_l$-$f_u$ [Hz]</th>
<th>Octave band filter $f_l$-$f_u$ [Hz]</th>
<th>Band no.</th>
<th>Center frequency $f_c$ [Hz]</th>
<th>3rd-octave band filter $f_l$-$f_u$ [Hz]</th>
<th>Octave band filter $f_l$-$f_u$ [Hz]</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>1.25</td>
<td>1.12 - 1.41</td>
<td></td>
<td>23</td>
<td>200</td>
<td>178 - 224</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.6</td>
<td>1.41 - 1.78</td>
<td>1.41 - 2.82</td>
<td>24</td>
<td>250</td>
<td>224 - 282</td>
<td>178 - 355</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1.78 - 2.24</td>
<td>1.41 - 2.82</td>
<td>25</td>
<td>315</td>
<td>282 - 355</td>
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<td>2.5</td>
<td>2.24 - 2.82</td>
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<td>26</td>
<td>400</td>
<td>355 - 447</td>
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<td>3.15</td>
<td>2.82 - 3.55</td>
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<td>27</td>
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<td>447 - 562</td>
<td>355 - 708</td>
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<td>6</td>
<td>4</td>
<td>3.55 - 4.47</td>
<td>2.82 - 5.62</td>
<td>28</td>
<td>630</td>
<td>562 - 708</td>
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<td>708 - 1410</td>
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### Table 2 contd. Standardized center frequencies and upper and lower frequency limits of third-octave and octave band filters. Shading indicates octave bands

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<td>10</td>
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<td>1410 - 1780</td>
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<td>4000</td>
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<td>5000</td>
<td>4470 - 5620</td>
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<td>5620 - 7080</td>
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Table 2 contd.  Standardized center frequencies and upper and lower frequency limits of third-octave and octave band filters. Shading indicates octave bands

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<td>17</td>
<td>50</td>
<td>44.7 - 56.2</td>
<td></td>
<td>39</td>
</tr>
<tr>
<td>18</td>
<td>63</td>
<td>56.2 - 70.8</td>
<td>44.7 - 89.1</td>
<td>40</td>
</tr>
<tr>
<td>19</td>
<td>80</td>
<td>70.8 - 89.1</td>
<td></td>
<td>41</td>
</tr>
<tr>
<td>20</td>
<td>100</td>
<td>89.1 - 112</td>
<td></td>
<td>42</td>
</tr>
<tr>
<td>21</td>
<td>125</td>
<td>112 - 141</td>
<td>89.1 - 178</td>
<td>43</td>
</tr>
<tr>
<td>22</td>
<td>160</td>
<td>141 - 178</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Addition of Frequency Components

Figure 9 The same sound spectrum presented in narrow band, third octave bands, and octave bands. The bigger the bandwidth, the more frequency components that contribute to any band, giving higher levels. The logarithmic frequency axis causes the CRB filters to have the same apparent width per band over the entire spectrum, while CAB filters, with their fixed band width over the entire spectrum, appear to grow more dense as frequency increases.
Summation of the sound pressures of individual frequency components is carried out in the same way as for the summation of sound pressures from multiple sources.

\[
\tilde{p}_{tot}^2 = \sum_{n=1}^{N} \tilde{p}_n^2
\]

\[
L_{p_{tot}} = 10\log\sum_{n=1}^{N} 10^{L_{pn}/10}
\]

where the index \( n \) stands for individual frequencies or frequency bands, instead of distinct sources. The proof for each of these formulas is from Parseval’s relations for periodic and non-periodic functions, which in turn comes from Fourier analysis.
Signal Analysis and Measurement Techniques

4. Measurement Systems for Sound and Vibrations

- The nature of sound and vibrations to be measured can vary widely.

- Sound can be “noisy” (roar or hiss-like), like that from a heavily trafficked highway, while vibrations of a machine are often dominated by the rotational frequency and its multiples.

- A machine under constant loading gives off a stationary noise, while the noise at an airport tends to be intermittent.

- Moreover, the purpose of measurements varies. If it is merely a question of a noise disturbance survey in an industrial facility, then relatively simple, single-channel instruments are used.

- If the purpose, on the other hand, is to determine the mode shapes of a large structure, such as an airplane fuselage, then a larger measurement system with two or more channels is required.
Figure 10  The most common portable instrument for measurement and analysis of sound is a sound level meter, which can be found in various makes and models. It measures the total sound pressure level throughout the audible band, but is often also equipped with octave band and third-octave band filters for frequency band analysis. To imitate the sensitivity of human hearing to sound with different frequency contents, so called A, B, and C weighting filters can be used.
Signal Analysis and Measurement Techniques

The Measurement Process

Figure 11 Flow chart of a digital, single-channel measurement system. For multi-channel systems, the analog and digital conversions are synchronously controlled for simultaneous sampling of the measurement signals.
Signal Analysis and Measurement Techniques

Transducers convert measured quantities, such as sound pressure, into equivalent electrical signals. A harmonically varying sound pressure induces a harmonic electrical signal at the same frequency. For a certain microphone, the output voltage $U(t) \ [\text{V}]$ due to a certain sound pressure $p(t)$ is

$$U(t) = C \ p(t)$$

where the proportionality constant $C$ should be independent of amplitude and frequency, to the extent possible.
Signal Analysis and Measurement Techniques

A number of characteristics are common to all types of transducers:

**Sensitivity**: Indicates the ratio of electrical output to mechanical input. Example: A microphone’s sensitivity is given in mV/Pa.

**Frequency band**: Indicates the upper and lower frequency limits, between which the transducer sensitivity varies within a given (small) tolerance range.

**Dynamic range**: Indicates the upper and lower amplitude limits between which the transducer sensitivity varies within a given (small) tolerance range. The dynamic range is commonly given in dB with respect to a reference value. The lower dynamic boundary is often determined by the transducer’s electrical noise, and the upper boundary by when the transducer is loaded beyond its mechanical linear region.
Signal Analysis and Measurement Techniques

Microphones

For acoustic measurements with high demands on the precision, condenser microphones are used. Condenser microphones consist of a thin metal membrane called a diaphragm, separated from an opposing “backplate” electrode by an air gap. The diaphragm and backplate constitute the electrodes of a condenser which is polarized by an electrical charge. When the diaphragm vibrates, due to the sound pressure, the capacitance of the condenser varies and an electrical output signal is generated. That electrical output signal is proportional to the sound pressure.
A small microphone has a higher upper frequency limit, but a lower sensitivity. Measurement microphones are therefore made in various sizes, and with built-in corrections for different types of sound fields. Microphone sizes are given in inches; typical sizes are 1", ½", ¼” and 1/8”. A ½” microphone is standard in most measurement situations.
Signal Analysis and Measurement Techniques

Figure 12 The electrode charges can be brought about in two ways. For externally polarized microphones, an external voltage is applied across the diaphragm and backplate. Pre-polarized microphones are charged by means of a thin electrical material that is placed on the backplate. That solution is usually more expensive, but nevertheless preferred for portable instruments, since it avoids the complications inherent in requiring external electrical voltage. (Source: Brüel & Kjær, Measurement Microphones)
The different types of microphones are:

**Free Field**

*Free field microphones* are intended for use in direct fields and should therefore be directed towards the sound source. They have built-in corrections that compensate the microphones influence on the sound field.

**Pressure**

*Pressure microphones* are mainly intended for calibrations in small cavities and for mounting positions flush with walls and the like. These do not compensate for their own affect on the sound field. They measure the actual sound pressure on the microphone diaphragm.
Signal Analysis and Measurement Techniques

Diffuse field

*Diffuse field microphones* are used in diffuse sound fields, i.e., they should have a flat sensitivity curve as a function of frequency for sound that falls in from all directions.

The microphone capsule is directly connected to a *pre-amplifier*. Its main task is to convert the microphone’s high output impedance to a low one, permitting connection to long cables or to a measurement system with relatively low input impedance.
Table 3 Data for some types and sizes of measurement microphones.

<table>
<thead>
<tr>
<th>Microphone</th>
<th>B&amp;K 4145</th>
<th>B&amp;K 4165</th>
<th>B&amp;K 4135</th>
<th>B&amp;K 4138</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter</td>
<td>1&quot; (25.4 mm)</td>
<td>½&quot; (12.7 mm)</td>
<td>¼&quot; (6.35 mm)</td>
<td>1/8&quot; (3.2 mm)</td>
</tr>
<tr>
<td>Type of sound field</td>
<td>Free field</td>
<td>Free field</td>
<td>Free and diffuse field</td>
<td>Pressure and diffuse field</td>
</tr>
<tr>
<td>Frequency range (± 2 dB)</td>
<td>2.6 Hz - 18 kHz</td>
<td>2.6 Hz - 20 kHz</td>
<td>4 Hz - 100 kHz</td>
<td>6.5 Hz - 140 kHz</td>
</tr>
<tr>
<td>Sensitivity [mV/Pa]</td>
<td>50</td>
<td>50</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Dynamic range [dB] (with recommended Pre-amp)</td>
<td>11 - 146</td>
<td>15 - 146</td>
<td>36 - 164</td>
<td>55 - 168</td>
</tr>
</tbody>
</table>
Piezoelectric accelerometers are the most commonly used vibration transducers. In some measurement situations, however, other types may be preferred, such as optical and inductive transducers or strain gauges. The construction of a piezoelectric accelerometer of the compression type is seen below.

**Compression Type**

![Figure 13](image)

Figure 13 Fundamental construction of a piezoelectric accelerometer of the compression variety.
Figure 14 The piezoelectric materials usually are artificially polarized ceramics that give rise to a charge, \( q \), when subjected to a) compression, or b) shear. The accelerometers that are based on the shear principle can be made less sensitive to other types of deformations, as for example, those caused by temperature variations. (Source: Brüel & Kjær, Piezoelectric Accelerometers and Vibration Preamplifiers)
Signal Analysis and Measurement Techniques

Selection of Accelerometers

There are primarily three characteristics of interest:

- sensitivity, given in charge per unit acceleration [pC/ms-2]
- internal resonance frequency
- the accelerometer’s total mass

The sensitivity and resonance frequency are strongly dependent on the mass.

Accelerometers are therefore available with masses ranging from one gram, for very high vibration levels (shocks), up to 500 grams, intended for very low levels.
Figure 15 The sensitivity of an accelerometer depends primarily on the seismic mass and the properties of the piezoelectric material. A larger accelerometer normally has a greater sensitivity. The seismic mass and the piezoelectric element constitute a mass-spring system with a resonance frequency $f_r$ at which the sensitivity increases dramatically. As a rule of thumb, it can be said that the useful frequency range is below $f_r/3$. 
### Table 4  Data for some types and sizes of accelerometers.

<table>
<thead>
<tr>
<th>Accelerometer</th>
<th>B&amp;K 8306</th>
<th>B&amp;K 4370</th>
<th>B&amp;K 4367</th>
<th>B&amp;K 4344</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass [g]</td>
<td>500</td>
<td>54</td>
<td>13</td>
<td>2</td>
</tr>
<tr>
<td>Type</td>
<td>Compression</td>
<td>Shear</td>
<td>Shear</td>
<td>Compression</td>
</tr>
<tr>
<td>Internal resonance frequency [Hz]</td>
<td>4500</td>
<td>18000</td>
<td>32000</td>
<td>70000</td>
</tr>
<tr>
<td>Frequency range [Hz]</td>
<td>0.06 - 1250</td>
<td>0.2 - 6000</td>
<td>0.2 – 10600</td>
<td>1 - 21000</td>
</tr>
<tr>
<td>Sensitivity [pC/ms-2]</td>
<td>1000</td>
<td>10</td>
<td>2</td>
<td>0.25</td>
</tr>
</tbody>
</table>
Signal Analysis and Measurement Techniques

• Piezoelectric accelerometers are connected to preamplifiers, the primary purpose of which is to match the high output impedance of the accelerometer to the low input impedance of the measurement system.

• Typically, a charge amplifier is used; despite its name, it does not amplify charge, but rather gives an output voltage proportional to the accelerometer’s charge.

• The particular advantage of a charge amplifier is that the length of the cable between the accelerometer and the amplifier can be varied without altering the sensitivity.
Advanced charge amplifiers even have other functions built in:

(i) Amplification of the signal to a level suited to the measurement system.

(ii) Integration of the acceleration signal to velocity and displacement. Usually, velocity signals are preferred, especially for acoustic measurements.

(iii) Low and high pass filters for damping of frequency components outside of the band to be analyzed, as, for example, around internal resonances and resonances due to the fastening to the measurement object.

It can be an advantage for some applications to use accelerometers with built-in pre-amplifiers. Such accelerometers are normally less sensitive to electrically induced noise in cables and thereby permit the use of long and inexpensive cables.
Mounting of accelerometers
There are three aspects, that must be considered when mounting accelerometers to measurement objects:

Figure 16 Measurement direction and placement
(Sketch: Brüel & Kjær, course material.)
Figure 17  The larger accelerometers have a higher sensitivity and give, therefore, stronger output signals. The accelerometer mass, however, also loads the measurement object itself, modifying its amplitude, especially at high frequencies. A coarse rule of thumb, for measurements on rigid bodies, is that the accelerometer mass should be less than a tenth of the mass of the measurement object. (Sketch: Brüel & Kjær, course material.)
Attachment to the measurement object

Figure 18 (1) Mounting with a threaded steel stud to a smoothly ground surface gives a high resonance frequency. (2) To avoid grounding loops in the measurement system, electrically isolating studs and discs can be used. (3) Wax between the accelerometer and the object is a common method that gives a surprisingly high resonance frequency, but which limits the temperature during measurements to a maximum of 40-50°C. (4) The accelerometer can be threaded onto special discs that are cemented to the measurement object. (5) Fastening magnets are practical on magnetic surfaces, but give a typical resonance frequency of about 7 kHz, which limits their useful range to about 2 kHz. (6) With the accelerometer placed on a hand-held measurement tip, a low resonance frequency is obtained, so that this method should not be used above 500-1000 Hz. (Sketch: Brüel & Kjær, Mechanical Vibration and Shock Measurements.)
Signal Analysis and Measurement Techniques

Calibration of transducers and measurement systems

The entire measurement chain is calibrated simultaneously. The transducer is subjected to a known vibration or sound signal, and the system’s output display is adjusted to show the correct value. That is the most common and convenient method to calibrate instrumentation prior to a measurement.

Figure 19 Piston phones and microphone calibrators are battery driven portable units for the calibration of acoustic measurement systems. Microphones are put into one end of a small cavity. In the other end, there is a piston or a membrane that is excited by an electrical motor or a piezoelectric element. The excitation gives a tone with a known frequency and sound level. (Source: Brüel & Kjær.)
Figure 20  To calibrate a vibration measurement system, there are portable, battery-driven electrodynamic vibrators, that generate vibrations of a known frequency and amplitude. (Photo: Brüel & Kjær.)
Signal Analysis and Measurement Techniques

Thank you